Imperial College London

Lecture 8

Frequency Responses of a System

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In this lecture, I will cover amplitude and phase responses of a system in some details. What I will attempt to do is to explain how would one be able to obtain the frequency response from the transfer function of a system. I will then show how once you have the amplitude and phase responses, you can predict the output signal for a given input signal if it is a sinusoidal.

Transfer Function H(s) vs Frequency Response H(jω)

Laplace Transform

$$\mathcal{L}\{x(t)\} = X(s) = \int_0^\infty x(t)e^{-st}dt$$

$s = j\omega$ →

Fourier Transform

$$\mathcal{F}\{x(t)\} = X(j\omega) = \int_0^\infty x(t)e^{-j\omega t}dt$$

Transfer Function

$$H(s)$$

$s = j\omega$ →

Frequency Response

$$H(j\omega)$$

- H(jω) is often expressed in polar form:

Frequency Response

$$H(s)\big|_{s=j\omega} = H(j\omega) = |H(j\omega)|\, e^{j\angle H(j\omega)}$$

Amplitude Response: $|H(j\omega)|$

Phase Response: $\angle H(j\omega)$

$\cos \omega t$ → $H(j\omega)$ → $|H(j\omega)|\cos[\omega t + \angle H(j\omega)]$

L4.8 p423

Let us remind ourself the definitions of Laplace and Fourier transforms. Assume the signal is causal (i.e. only starts at t=0), then from the above definition, it is clear that Fourier transform of a signal can be obtained if we substitute s = jω.

While this is true for signal, something similar is true for a system. A system in s-doamin is characterizsed by its transfer function (H(s) = output Y(s) / input X(s).

The frequency response H(jω) is a function that relates the output response to a sinusoidal input at frequency ω. They are therefore, not surprisingly, related. In fact the frequency response of a system is simply its transfer function as evaluated by substituting s = jω.

The frequency response H(jω) is in general is complex, with real and imaginary parts. This is often more useful and intuitive when expressed in polar coordinate. That is, we can separate H(jω) into its magnitude (called amplitude response) and its phase component (called phase response).

$$H(s)\big|_{s=j\omega} = H(j\omega) = |H(j\omega)|\, e^{j\angle H(j\omega)}$$

$|H(j\omega)|$ is the amplitude response.

$\angle H(j\omega)$ is the phase response.

Note that $e^{j\angle H(j\omega)}$ has a magnitude of 1 and a phase of $\angle H(j\omega)$.

Frequency Response Example (1)

- Find the frequency response of a system with transfer function:

$$H(s)=\frac{s+0.1}{s+5}$$

- Then find the amplitude and phase response $y(t)$ for inputs:
  (i) $x(t)=\cos 2t$ and (ii) $x(t)=\cos(10t-50°)$

- Substitute $s=j\omega$

$$H(j\omega)=\frac{j\omega+0.1}{j\omega+5}$$

$$|H(j\omega)|=\frac{\sqrt{\omega^2+0.01}}{\sqrt{\omega^2+25}} \quad \text{and} \quad \angle H(j\omega)=\Phi(j\omega)=\tan^{-1}\left(\frac{\omega}{0.1}\right)-\tan^{-1}\left(\frac{\omega}{5}\right)$$

Now let us apply what is explained in the previous slides to some examples. Given that the transfer fucntion of a system is:

$$H(s)=\frac{s+0.1}{s+5}$$

We want to find the amplitude response and phase response of the system to two sinusoidal signals at the input:

(i) $x(t)=\cos 2t$ and (ii) $x(t)=\cos(10t-50°)$

The first signal is a simple cosine wave. The second is a cosine signal with a phase shift of 50 degrees.

First we substitute s = jω into H(s) to obtain an expression of the frequency response. Note that the numerator and the denomator are both complex.

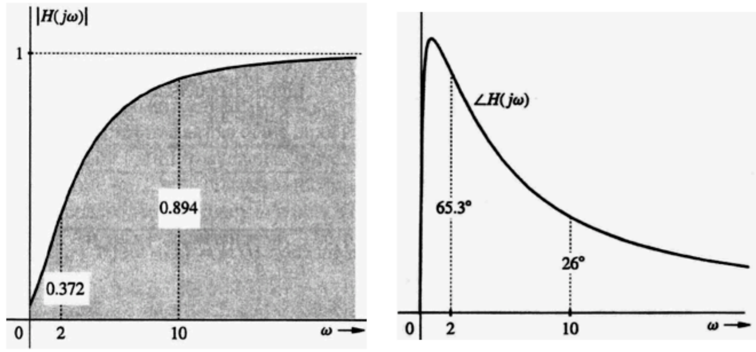
To obtain the amplitude response, we take the absolute value of H(jω). To do this, we evaluate the magnitude of the numerator and the denominator separately.

To obtain the phase response, we take the arctan of the numerator, and subtract from it the arctan of the denominator. (Angle of a complex number expressed as a vector is something you may not be familiar with. Don't worry. I include this here for completeness. For this course, I want to focus on amplitude response, and include phase response for information only.)

The phase of the numerator is therefore tan^{-1} (Imaginary part / real part) = tan^{-1} (ω/0.1).

Frequency Response Example (2)

$$|H(j\omega)|=\frac{\sqrt{\omega^2+0.01}}{\sqrt{\omega^2+25}} \qquad \Phi(j\omega)=\angle H(j\omega)=\tan^{-1}\left(\frac{\omega}{0.1}\right)-\tan^{-1}\left(\frac{\omega}{5}\right)$$



for input $x(t)=\cos 2t$ and $x(t)=\cos(10t-50°)$

Now let us consider our input signals:

(i) $x(t)=\cos 2t$ and (ii) $x(t)=\cos(10t-50°)$

The two signals have frequency at 2 and 10 (rad/sec).

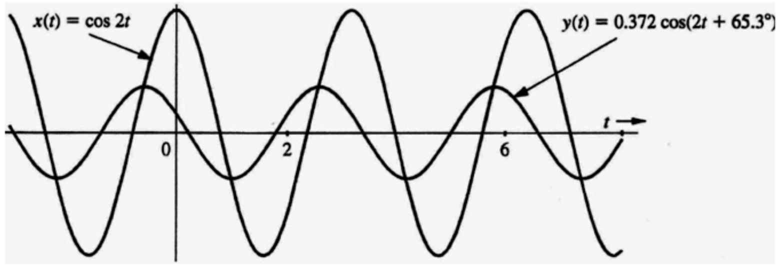
If we now plot the amplitude response $|H(j\omega)|$ and phase response $\angle H(j\omega)$ we get the two plots as shown. (These can easily be obtained using Matlab.) We just read off the values at the two frequencies from the two graphs!

Frequency Response Example (3)

- For input $x(t)=\cos 2t$, we have:

$$|H(j2)| = \frac{\sqrt{2^2+0.01}}{\sqrt{2^2+25}} = 0.372 \qquad \Phi(j2) = \tan^{-1}\left(\frac{2}{0.1}\right) - \tan^{-1}\left(\frac{2}{5}\right) = 65.3^\circ$$

- Therefore $y(t) = 0.372\cos(2t + 65.3^\circ)$



Instead of reading the values off the graphs (assume that the plots are not available), you can simply calculate the amplitude gain and phase gain at the two frequencies. For $\omega = 2$, $|H(j\omega)| = 0.372$, and the phase at this frequency is 65.3 degrees.

How do we interpret this results? What it means is the following:

The input cosine signal at frequency 2 rad/sec will have its amplitude reduced from 1v to 0.372v. Furthermore, there will be a phase shift of +65.3° added to the phase of the original cosine signal.

Frequency Response Example (4)

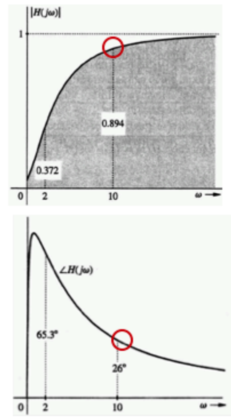
- For input $x(t)=\cos(10t-50^\circ)$, we will use the amplitude and phase response curves directly:

$$|H(j10)| = 0.894$$

$$\Phi(j10) = \angle H(j10) = 26^\circ$$

- Therefore

$$y(t) = 0.894\cos(10t - 50^\circ + 26^\circ) = 0.894\cos(10t + 24^\circ)$$



Similarly we can work out what happens for the second signals. I will leave you to figure it out for yourself!

Frequency Response of delay of T sec

- H(s) of an ideal T sec delay is:

  $H(s) = e^{-sT}$ (Time-shifting property)

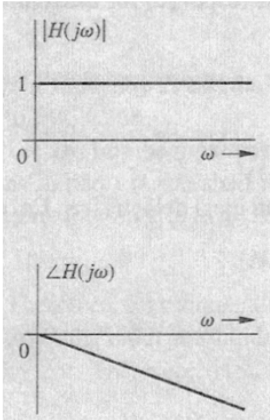
- Therefore

  $|H(j\omega)| = |e^{-j\omega T}| = 1$ and $\Phi(j\omega) = -\omega T$

- That is, delaying a signal by T has **no effect** on its amplitude.
- It results in a linear phase shift (with frequency), and a gradient of –T.
- The quantity:

  $$-\frac{d\Phi(\omega)}{d\omega} = \tau_g = \text{T}$$

  is known as **Group Delay**.



In the next three slides, I want to explore the frequency response of three important system operations:

1. Time delay (by T sec)
2. Differentiator (d/dt)
3. Integrator (∫ dt)

First time delay. The transfer function of a pure time delay of T second is:

$$H(s) = e^{-sT}$$

This has been proven in Lecture 7, slide 21. It is known as the time-shifting property of Laplace transform and is one of the few facts that is worth remembering.

Therefore, the magnitude of H(jω) is 1 and the phase of H(jω) is –ωT.

The important key point to takeaway is that time delay does NOT change the amplitude of a signal (obvious through intuition). However it changes the phase.

Frequency Response of an ideal differentiator

- H(s) of an ideal differentiator is:

  $H(s) = s$ and $H(j\omega) = j\omega = \omega e^{j\pi/2}$

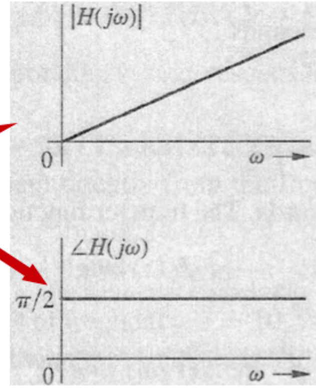
- Therefore

  $|H(j\omega)| = \omega$ and $\angle H(j\omega) = \frac{\pi}{2}$

- This agrees with:

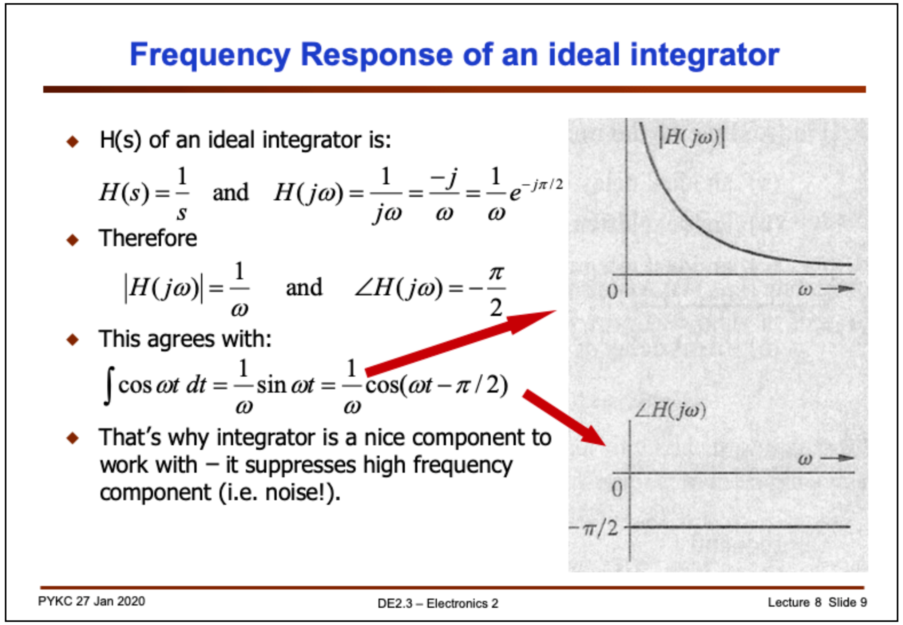
  $$\frac{d}{dt}(\cos\omega t) = -\omega\sin\omega t = \omega\cos(\omega t + \pi/2)$$

- That's why differentiator is **not** a nice component to work with – it **amplifies high frequency** component (i.e. noise!).

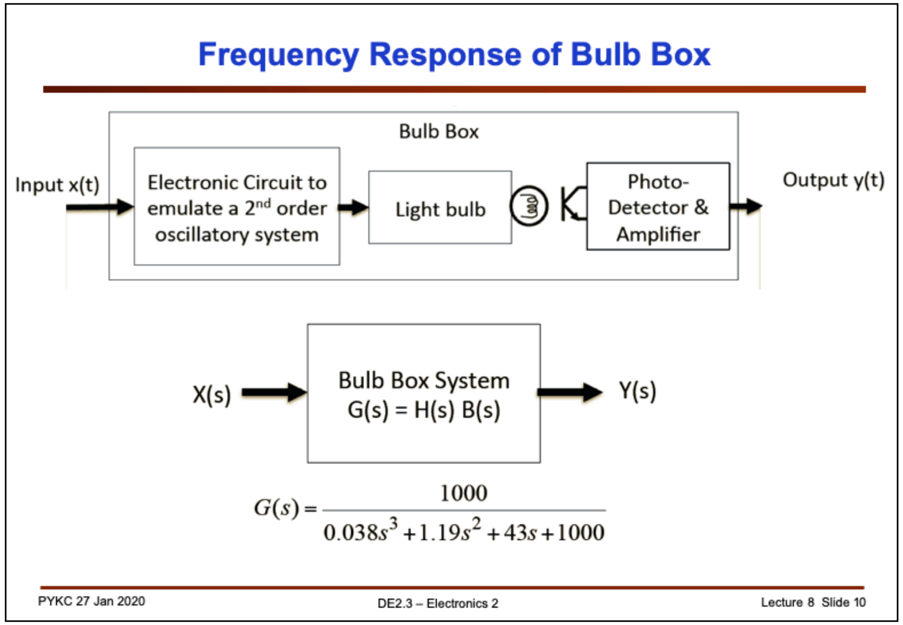


Now let us consider the ideal differentiator (d/dt). The transfer function of a differentiator H(jω) = jω. Therefore the amplitude response |H(jω)| = ω. The phase is a constant 90 degrees or π/2.

The takeaway message here is that differentiator is a highpass filter. It AMPLIFIES high frequency signals. Since noise in a signal tends to reside in high frequency components, differentiators usually produces an even nosier signal at the output.



Finally we can apply the same principle to derive the frequency response of an integrator. Unlike a differentiator, an integrator has a lowpass filter effect. It therefore suppresses high frequency components and therefore suppress noise.



Let us now apply what we have learned in this Lecture to Lab 2 experiment using the Bulb Box. The box has an electronic circuit which behaves like a second-order system with a natural frequency of 5Hz and a very low damping ratio (i.e. highly oscillatory). The output of this circuit drives the light-bulb and photo-diode circuit to produce an output depending on the light intensity.

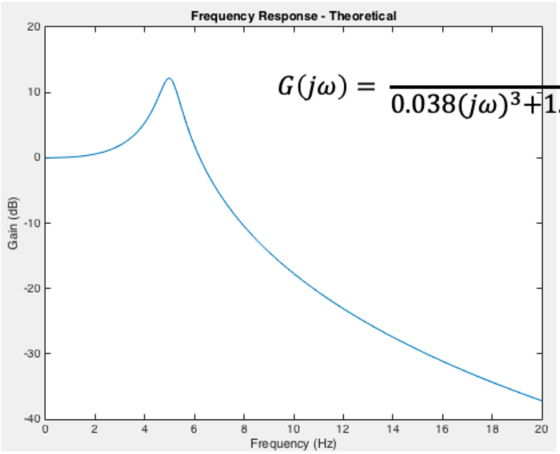
We can model this system as shown in the slide mathematically as a transfer function G(s) in the complex frequency (s) domain:

$$G(s)=\frac{1000}{0.038s^3+1.19s^2+43s+1000}$$

To find the frequency response of the bulb box, we simply evaluate G(s) at s=jω into this equation:

$$G(s)|_{s=j\omega}=\frac{1000}{0.038(j\omega)^3+1.19(j\omega)^2+43j\omega+1000}$$

Theoretical Frequency Response of Bulb Box



$$G(j\omega) = \frac{1000}{0.038(j\omega)^3 + 1.19(j\omega)^2 + 43j\omega + 1000}$$

Plotting the magnitude $|G(j\omega)|$ in dB vs frequency is the SAME as plotting the amplitude spectrum of the system.

For the Bulb box, the frequency response is peaky at 5Hz as you would expect because this is the resonant frequency of the system – that is, the system ”likes” this frequency! The voltage gain at this frequency is around 12dB or a gain of around x20.

The system behaves like a low pass filter because at high frequency, the output is strongly suppressed. Beyond 20Hz, the gain drops to around -40dB (or an attenuation of 100).

Remember that frequency response of a system is a measure of its response to sinusoidal input AT STEADY STATE – that is, after all the transient has died down. Furthermore, because our Bulb Box is non-linear. That means the output voltage is not a linear function of the input. In general, all systems are not perfectly linear. We often “pretend” that the system is linear by operating over a small range of signal.